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Differential Game of Many Pursuers and Evaders with Integral Constraints on a Cylinder

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Abstract

We consider a simple motion pursuit differential game of m pursuers and k evaders on a cylinder in $\mathbb{R}^3$. Pursuit is said to be completed if the position of each evader coincides with the position of a pursuer at some finite time. We reduce the differential game to a differential game of m groups of countably many pursuers and k groups of countably many evaders in $\mathbb{R}^2$ where all the players from each group are controlled by one control parameter subject to integral constraint. We prove that if the total resource of the pursuers is greater than the total resource of the evaders, then pursuit can be completed. We construct strategies for the pursuers.

Keywords: Differential game on cylinder; integral constraint; many pursuers; evaders; strategy.

1 Introduction

Differential games involving many players are the extension of differential games involving single pursuer and evader (see, for example, [6, 13]). Simple motion differential games with many pursuers were intensively studied by many researchers such as [1] and [16]. Interesting results were obtained in the study of differential games with state constraints. Melikyan and Ovakimyan [10, 11] applied the Isaacs method to study a simple motion pursuit differential games on two dimensional Riemann manifolds. The payoff in the game is the minimum distance between the players in a semi-infinite interval of the time of movements. The researchers established the necessary conditions for the singular surface that are optimal and derived the equations of the singular paths. Azamov and Kuchkarov [2] examined the problem of evasion of one evader from many pursuers on a hypersurface in $\mathbb{R}^n$. Ivanov [7] studied the differential game on convex compact set. All players involved in the game have equal dynamic possibilities and during the game, all players may not leave a given compact subset subset N of $\mathbb{R}^n$. Ivanov obtained necessary and sufficient condition of the game.

Simple motion pursuit and evasion differential games studied by Kuchkarov [8] occur on a ball of the Riemann manifold, and all players have the same dynamical possibilities. All the pursuers move on the ball, and with respect to the motion of the evader two cases were considered: in the first case, the evader moves along the sphere of the ball and in the second case, the evader also moves on the ball. For each case, sufficient conditions of both completion of pursuit and evasion were obtained. Kuchkarov et al. [9] studied a differential game of many pursuers and one evader on a cylinder when controls of players are subject to geometric constraints. Two cases were considered: in the first case the pursuers move without phase constraints and in the second case, the pursuers move on the surface of the cylinder. The researchers obtained necessary and sufficient conditions to complete the pursuit in both cases. It is shown that in the second case, the pursuit game on a cylinder is equivalent to a differential game on the plane.

Petrov and Shchelchkov [14, 15] examined games of many pursuers and evaders where all the evaders are controlled by one control parameter, but pursuers have different control parameters. By definition, game is said to be completed if the state of a pursuer coincides with that of an evader at some time. In these papers, sufficient conditions of completion of pursuit were obtained. Ramana and Kothari [17] studied pursuit-evasion games of high speed evader involving many pursuers and a single evader in an open domain. The researchers discussed the required formation and capture strategy for a group of pursuers.

There are many works devoted to differential games of many players with integral constraints. Satimov et al. [18] considered a linear differential game. The researchers obtained sufficient conditions for the completion of pursuit in the game. Ibragimov and Hasim [5] considered pursuit and evasion differential game problems with countably many pursuers in Hilbert space. The researchers found the sufficient condition that the total resource of the pursuers is less than that of the evader for the evasion problem while the total resource of the pursuers greater than that of the evader for the pursuit problem. Ferrara et al. [3] studied a simple motion evasion differential game of many pursuers and evaders with integral constraints. The researchers obtained sufficient condition of evasion and constructed the strategies for evaders. This condition is sharp since if this condition is not satisfied, then it was shown by Ibragimov and Satimov [4] that pursuit can be completed.

In the present paper, we study simple motion differential game of m pursuers and k evaders on a cylinder in $\mathbb{R}^3$ with integral constraints on the control functions of players. Pursuit is said to be completed if the position of each evader coincides with the position of a pursuer. We reduce the differential game to a differential game of m groups of countably many pursuers and k groups of countably many evaders in $\mathbb{R}^2$. All the players from each group are controlled by one control parameter subject to an integral constraint. We show that if the total resource of the pursuers is greater than the total resource of the evaders, then pursuit can be completed. We construct strategies for the pursuers.

2 Statement of Problem

Let

$$M = \left\{x = \left(x_{(1)}, x_{(2)}, x_{(3)}\right) \in \mathbb{R}^3 \mid x_{(1)}^2 + x_{(2)}^2 = b^2,\ x_{(3)} \in \mathbb{R}\right\},$$

that is, M is the surface of a two-dimensional cylinder in $\mathbb{R}^3$ and $M_s, s \in M$, be the tangent plane to the surface of the cylinder M at point s. The movements of the pursuers x_i and the evaders y_j on the surface of cylinder M are described by the following equations:

$$\begin{aligned} &\dot{x}_i = u_i, \quad x_i(t_0) = x_{i0}, \\ &\dot{y}_j = v_j, \quad y_j(t_0) = y_{j0}, \quad y_{j0} \neq x_{i0}, \end{aligned} \tag{1}$$

where $x_i, y_j \in M, u_i \in M_{x_i}, v_j \in M_{y_j}$; u_i and v_j are control parameters of the players $x_i,\ i = 1, 2, \dots, m$, and $y_j,\ j = 1, 2, \dots, k$, respectively.

Definition 2.1. *A measurable function $u_i(t),\ t \geq t_0,$ is called admissible control of pursuer x_i, if*

$$\int_0^\infty |u_i(s)|^2 ds \leq \rho_i^2, \tag{2}$$

and along the solution $x_i(\cdot)$ of the initial value problem $\dot{x}_i(t) = u_i(t),\ x_i(t_0) = x_{i0}$, the inclusion

$$u_i(t) \in M_{x_i(t)} \tag{3}$$

is satisfied, where $\rho_i,\ i = 1, 2, \dots, m$, is a given positive number.

Definition 2.2. *A measurable function $v_j(t),\ t \geq t_0,$ is called admissible control of evader y_j, if*

$$\int_0^\infty |v_j(s)|^2 ds \leq \sigma_j^2, \tag{4}$$

and along the solution $y_j(\cdot)$ of the initial value problem $\dot{y}_j(t) = v_j(t),\ y_j(t_0) = y_{j0}$, the inclusion

$$v_j(t) \in M_{y_j(t)} \tag{5}$$

is satisfied, where $\sigma_j,\ j = 1, 2, \dots, k$, is a given positive number.

Definition 2.3. *A function $U_i(t, x_i, y_1, \dots, y_k, v_1, \dots, v_k)$*

$$U_i : [t_0, \infty) \times M^{k+1} \times \mathbb{R}^{3k} \to M_{x_i},\ \ i = 1, 2, \dots m,$$

is called strategy of pursuer x_i if for any admissible control $v_j(t),\ t \geq 0$, of evader y_j, the initial value problem

$$\begin{aligned} &\dot{x}_i = U_i(t, x_i, y_1, \dots, y_k, v_1, \dots, v_k),\ \ x_i(t_0) = x_{i0}, \\ &\dot{y}_j = v_j(t),\ \ y_j(t_0) = y_{j0},\ \ j = 1, 2, \dots, k, \end{aligned} \tag{6}$$

has a unique absolutely continuous solution $(x_i(t), y_1(t), \ldots, y_k(t))$, $x_i(t), y_j(t) \in M$, and along the solution, the inclusion

$$U_i(t, x_i(t), y_1(t), \ldots, y_k(t), v_1(t), \ldots, v_k(t)) \in M_{x_i(t)} \tag{7}$$

and the inequality

$$\int_0^\infty |U_i\,(t, x_i(t), y_1(t), \ldots, y_k(t), v_1(t), \ldots, v_k(t))\,|^2 ds \leq \rho_i^2 \tag{8}$$

hold.

Definition 2.4. *We say that pursuit can be completed from the initial position $\{x_{10}, \ldots, x_{m0}, y_{10}, \ldots, y_{k0}\}$ for the time T in the game (1)–(7), if there exist strategies $U_i,\ i = 1, 2, \ldots, m$, of pursuers such that for any controls $v_1 = v_1(\cdot), \ldots, v_k = v_k(\cdot)$ of the evaders and for each $j \in \{1, 2, \ldots, k\}$, the equality $x_i(t_j) = y_j(t_j)$ holds for some $i \in \{1, 2, \ldots, k\}$ at some time $t_j \in [0, T]$.*

Problem *Find a sufficient condition for completing pursuit in the game (1)–(7) and construct strategies for the pursuers.*

3 Main Results

We formulate the main result of this paper. We first reduce the game (1)–(7) on the surface of the cylinder M to a specific game in $\mathbb{R}^2$ by unfolding M in $\mathbb{R}^2$ ([12]). Such unfolding is presented by the multivalued mapping that is inverse to the universal covering $\pi : \mathbb{R}^2 \to M$, which is local isometry. If $z \in M$, then the set of its preimages $\pi^{-1}(z)$ consists of the class of denumerable points, $z_1, z_2, ... \in \mathbb{R}^2$ equivalent to each other. Now the game (1)–(7) is reduced to a game in $\mathbb{R}^2$ in which m groups of countably many pursuers, $\pi^{-1}(x_i) = \{x_i^1, x_i^2, ...\},\ i = 1, 2, \ldots, m$, pursue k groups of evaders, $\pi^{-1}(y_j) = \{y_j^1, y_j^2, ...\},\ j = 1, 2, \ldots, k$.

The movements of the m groups of pursuers and k groups of evaders are described by the following equations:

$$\begin{aligned} &\dot{x}_i^a = u_i, \quad x_i^a(t_0) = x_{i0}^a, \quad i = 1, 2, \ldots, m;\ a = 1, 2, \ldots, \\ &\dot{y}_j^b = v_j, \quad y_j^b(t_0) = y_{j0}^b, \quad j = 1, 2, \ldots, k;\ b = 1, 2, \ldots, \end{aligned} \tag{9}$$

where $x_i^a, y_j^b, u_i, v_j \in \mathbb{R}^2,\ x_{i0}^a \neq y_{j0}^b$ for all $i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, k,\ a, b \in \{1, 2, ...\},\ u_i$ and v_j are control parameters of the m groups of pursuers and the k groups of evaders, respectively, which satisfy the constraints

$$\begin{aligned} &\int_{t_0}^\infty |u_i(s)|^2 ds \leq \rho_i^2, \quad i = 1, 2, \ldots, m, \\ &\int_{t_0}^\infty |v_j(s)|^2 ds \leq \sigma_j^2, \quad j = 1, 2, \ldots, k. \end{aligned} \tag{10}$$

All players of each group are controlled by the same control parameter. Consider the differential game (9)–(10) and choose one pursuer and one evader from each group. For short, we write the movements and the constraints of the pursuers and evaders without superscripts, as $x_i,\ i = 1, 2, \ldots, m$ and $y_j,\ j = 1, 2, \ldots, k$, respectively.

3.1 Differential Game of One Pursuer and One Evader

We first study an auxiliary differential game involving one pursuer x and one evader y. The movements of the pursuer and the evader are described by the following equations:

$$\dot{x} = u, \ x(t_0) = x_0, \ \dot{y} = v, \ y(t_0) = y_0, \tag{11}$$

where $x, y \in \mathbb{R}^2$, u is control parameter of pursuer x and v is that of evader y. The following are integral constraints

$$\int_{t_0}^{\infty} |u(s)|^2 ds \le \rho^2, \quad \int_{t_0}^{\infty} |v(s)|^2 ds \le \sigma^2,$$

where ρ and σ are positive numbers. Pursuit is completed if $x(t') = y(t')$ at some time $t' \ge t_0$.

Let $v(t), t \ge t_0$, be an arbitrary control of evader. Set

$$u(t) = U(t, x_0, y_0, v) = \frac{y_0 - x_0}{\theta} + v(t), \ \ t \ge t_0, \tag{12}$$

where θ is a positive number to be defined in the following lemma.

Lemma 3.1. *Let the pursuer use strategy (12).*

(i) If $\rho > \sigma$, then pursuit can be completed in the game (11) for the time θ, that satisfy the condition

$$\theta \ge \frac{|y_0 - x_0|^2}{(\rho - \sigma)^2}, \tag{13}$$

and, moreover, at the time θ,

$$\rho^2(\theta) \ge \rho^2 - \sigma^2 - \frac{1}{\theta}\left(|y_0 - x_0|^2 + 2\sigma\sqrt{\theta}|y_0 - x_0|\right). \tag{14}$$

(ii) If $\rho \le \sigma$, then, for any $\theta > 0$, either $x(\theta) = y(\theta)$ or

$$\sigma^2(\theta) \le \sigma^2 - \rho^2 + \frac{1}{\theta}\left(|y_0 - x_0|^2 + 2\sigma\sqrt{\theta}|y_0 - x_0|\right). \tag{15}$$

Proof. Let

$$\rho^2(t) = \rho^2 - \int_{t_0}^{t} |u(s)|^2 ds, \ \ \sigma^2(t) = \sigma^2 - \int_{t_0}^{t} |v(s)|^2 ds. \tag{16}$$

First, we prove item (i) of Lemma 3.1. Let $\rho > \sigma$. Strategy (12) is admissible since by the Cauchy-Schwartz inequality, we have

$$(y_0 - x_0)\int_{t_0}^{\theta} v(t)dt \le |y_0 - x_0| \int_{t_0}^{\theta} |v(t)|dt \le |y_0 - x_0|\sigma\sqrt{\theta}. \tag{17}$$

Then by (12) we have

$$\int_{t_0}^{\theta} |u(t)|^2 dt = \frac{1}{\theta}|y_0 - x_0|^2 + \frac{2}{\theta}(y_0 - x_0)\int_{t_0}^{\theta} v(t)dt + \int_{t_0}^{\theta} |v(t)|^2 dt.$$

Using (17), we obtain

$$\int_{t_0}^{\theta} |u(t)|^2 dt \le \frac{1}{\theta}\left(|y_0 - x_0|^2 + 2\sigma\sqrt{\theta}|y_0 - x_0|\right) + \int_{t_0}^{\theta} |v(t)|^2 dt. \tag{18}$$

In view of (13), the inequality (18) implies that

$$\int_{t_0}^{\theta} |u(t)|^2 dt \le (\rho-\sigma)^2 + 2(\rho-\sigma)\sigma + \sigma^2 = \rho^2.$$

Hence, strategy (12) is admissible. In particular, from inequality (18) we get

$$\rho^2(\theta) = \rho^2 - \int_{t_0}^{\theta} |u(t)|^2 dt \ge \rho^2 - \sigma^2 - \frac{1}{\theta}\left(|y_0 - x_0|^2 + 2\sigma\sqrt{\theta}|y_0 - x_0|\right).$$

Thus, (14) holds.
Now, it can be shown that the equality $x(\theta) = y(\theta)$ is ensured

$$x(\theta) = x_0 + \int_{t_0}^{\theta} \left(\frac{y_0 - x_0}{\theta} + v(t)\right) dt = y_0 + \int_{t_0}^{\theta} v(t) dt = y(\theta). \tag{19}$$

Let now $\rho \le \sigma$ and the pursuer use strategy (12) on the time interval $[0, \theta]$. If $\int_{t_0}^{\theta} |u(t)|^2 dt \le \rho^2$, that is

$$\int_{t_0}^{\theta} |u(t)|^2 dt = \int_{t_0}^{\theta} \left|\frac{y_0 - x_0}{\theta} + v(t)\right|^2 dt \le \rho^2,$$

then, clearly, the strategy (12) is admissible. It is obvious that the equality $x(\theta) = y(\theta)$ is ensured similar to (19).

If $x(\theta) \ne y(\theta)$, then we must have

$$\int_{t_0}^{\theta} |u(t)|^2 dt = \int_{t_0}^{\theta} \left|\frac{y_0 - x_0}{\theta} + v(t)\right|^2 dt > \rho^2. \tag{20}$$

Then applying (18) to this inequality, we get

$$\int_{t_0}^{\theta} |v(t)|^2 dt > \rho^2 - \frac{1}{\theta}\left(|y_0 - x_0|^2 + 2\sigma\sqrt{\theta}|y_0 - x_0|\right), \tag{21}$$

and hence,

$$\sigma^2(\theta) = \sigma^2 - \int_{t_0}^{\theta} |v(t)|^2 dt < \sigma^2 - \rho^2 + \frac{1}{\theta}\left(|y_0 - x_0|^2 + 2\sigma\sqrt{\theta}|y_0 - x_0|\right). \tag{22}$$

The proof of the lemma is complete. □

The inequality (21) shows that though $x(\theta) \ne y(\theta)$, the pursuer forces to expend the evader's energy more than

$$\rho^2 - \frac{1}{\theta}\left(|y_0 - x_0|^2 + 2\sigma\sqrt{\theta}|y_0 - x_0|\right).$$

At the same time, using the strategy (12), the pursuer will spend all his energy by a time t_1,

$$\int_{t_0}^{t_1} |u(t)|^2 dt = \rho^2, \;\; t_0 < t_1 \le \theta. \tag{23}$$

Then, of course, the pursuer cannot move anymore and hence, $u(t) \equiv 0, \; t \ge t_1$.

3.2 Differential Game of Many Pursuers and Evaders

We now formulate the main theorem of the present paper.

Theorem 3.1. *If*

$$\rho_1^2 + \rho_2^2 + \cdots + \rho_m^2 > \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_k^2, \tag{24}$$

then pursuit can be completed in the game (9)–(10) for a finite time.

Proof. Denote

$$f_{ij}(\tau, \theta) = \frac{1}{\theta - \tau} \left(|y_j(\tau) - x_i(\tau)|^2 + 2\sigma\sqrt{\theta - \tau}|y_j(\tau) - x_i(\tau)| \right). \tag{25}$$

Consider two cases. Case 1: $\rho_m > \sigma_k$. Let $\theta_1 > t_0$ be any number that satisfy the inequalities

$$\rho_1^2 + \rho_2^2 + \cdots + \rho_m^2 > \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_k^2 + f_{mk}(t_0, \theta_1) \tag{26}$$

and

$$\theta_1 \geq \frac{|y_{m0} - x_{k0}|^2}{(\rho_m - \sigma_k)^2}. \tag{27}$$

We construct the strategies of pursuers on the time interval $[t_0, \theta_1]$.
Set

$$u_m(t) = \frac{y_{m0} - x_{k0}}{\theta_1} + v_k(t), \quad t_0 \leq t \leq \theta_1, \tag{28}$$

$$u_i(t) = 0, \quad t_0 \leq t \leq \theta_1, \quad i = 1, 2, \ldots, (m-1). \tag{29}$$

Equation (29) shows that all the pursuers $x_1, x_2, \ldots, x_{m-1}$ do not move on the time interval $[t_0, \theta_1]$. By Lemma 3.1 the pursuers x_m can capture the evader y_k at the time θ_1 and by inequality (14) we have

$$\rho_m^2(\theta_1) \geq \rho_m^2 - \sigma_k^2 - f_{mk}(t_0, \theta_1). \tag{30}$$

Then according to (26), we have

$$\rho_1^2 + \cdots + \rho_{m-1}^2 + \rho_m^2(\theta_1) \geq \sigma_1^2 + \cdots + \sigma_{k-1}^2. \tag{31}$$

For $t > \theta_1$, we consider the game of pursuers $x_1, \ldots, x_m$ with the energies $\rho_1^2, \ldots, \rho_{m-1}^2, \rho_m^2(\theta_1)$ and evaders $y_1, \ldots, y_{k-1}$ with the energies $\sigma_1^2, \ldots, \sigma_{k-1}^2$ under condition (31). Thus, at the time θ_1 the number of players is decreased by 1.

Case 2: $\rho_m \leq \sigma_k$. In this case we choose θ_1 from the condition (26) and let the pursuers apply strategies (28) and (29). Then either $x_m(\theta_1) = y_k(\theta_1)$ or $x_m(\theta_1) \neq y_k(\theta_1)$. If $x_m(\theta_1) = y_k(\theta_1)$, then evader y_k is captured, and for $t > \theta_1$, we consider game of pursuers $x_1, \ldots, x_{m-1}$ with the energies $\rho_1^2, \ldots, \rho_{m-1}^2$, and evaders $y_1, \ldots, y_{k-1}$ with the energies $\sigma_1^2, \ldots, \sigma_{k-1}^2$ under the condition

$$\rho_1^2 + \cdots + \rho_{m-1}^2 > \sigma_1^2 + \cdots + \sigma_{k-1}^2. \tag{32}$$

In this case, the number of players is decreased by 2.

If $x_m(\theta_1) \neq y_k(\theta_1)$, then by inequality (15) in Lemma 3.1, we have

$$\sigma_k^2(\theta_1) \leq \sigma_k^2 - \rho_m^2 + f_{mk}(0, \theta_1). \tag{33}$$

Combining (26) and (33) yields

$$\rho_1^2 + \cdots + \rho_{m-1}^2 > \sigma_1^2 + \cdots + \sigma_k^2(\theta_1). \tag{34}$$

For $t > \theta_1$, we consider the game of $x_1, \dots, x_{m-1}$ and $y_1, \dots, y_k$ with the condition (34). Thus, the number of players is decreased by 1.

Thus, the differential game of m pursuers and k evaders started at t_0 is reduced to a differential game of p pursuers and q evaders at the time θ_1, where $p = m$, $q = k - 1$ if $\rho_m > \sigma_k$ (Case 1); $p = m - 1$, $q = k - 1$ if $\rho_m \le \sigma_k$ and $x_m(\theta_1) = y_k(\theta_1)$ (Case 2); $p = m - 1$, $q = k$ if $\rho_m \le \sigma_k$ and $x_m(\theta_1) \neq y_k(\theta_1)$ (Case 2). Also, we have

$$\rho_1^2(\theta_1) + \cdots + \rho_p^2(\theta_1) > \sigma_1^2(\theta_1) + \cdots + \sigma_q^2(\theta_1), \tag{35}$$

where

$$\rho_i^2(t) = \rho_i^2 - \int_{t_0}^{t} |u_i(s)|^2 ds, \;\; \sigma_j^2(t) = \sigma_j^2 - \int_{t_0}^{t} |v_j(s)|^2 ds, \tag{36}$$

where $i = 1, ..., m$, $j = 1, ..., k$. It should be noted that $p + q \le m + k - 1$, that is, the number of players at the time θ_1 is decreased at least by 1.

Let now θ_2, $\theta_2 > \theta_1$, be an arbitrary fixed number satisfying the inequalities

$$\rho_1^2(\theta_1) + \cdots + \rho_p^2(\theta_1) > \sigma_1^2(\theta_1) + \cdots + \sigma_q^2(\theta_1) + f_{pq}(\theta_1, \theta_2) \tag{37}$$

$$\theta_2 - \theta_1 \ge \frac{|y_q(\theta_1) - x_p(\theta_1)|^2}{(\rho_p(\theta_1) - \sigma_q(\theta_1))^2}, \tag{38}$$

if $\rho_p(\theta_1) > \sigma_q(\theta_1)$, and any positive number that satisfy (37) if $\rho_p(\theta_1) \le \sigma_q(\theta_1)$.

Set

$$\begin{aligned} u_p(t) &= \frac{y_q(\theta_1) - x_p(\theta_1)}{\theta_2 - \theta_1} + v_q(t), \quad \theta_1 < t \le \theta_2, & (39)\\ u_i(t) &\equiv 0, \quad \theta_1 < t \le \theta_2, \quad i \in \{1, 2, \dots, m\} \backslash \{p\}. & (40) \end{aligned}$$

Equation in (39) means that all pursuers except for x_p do not move on the time interval $(\theta_1, \; \theta_2]$.

We conclude the following conditions by letting the pursuers use the strategies (39), (40) and applying the same cases above:

(i) If $\rho_p(\theta_1) > \sigma_q(\theta_1)$, then by Lemma 3.1, the equality $x_p(\theta_2) = y_q(\theta_2)$ is satisfied. Starting from the time θ_2 we will consider the pursuit game involving p pursuers and $(q-1)$ evaders under the condition

$$\rho_1^2(\theta_2) + \cdots + \rho_p^2(\theta_2) > \sigma_1^2(\theta_2) + \cdots + \sigma_{q-1}^2(\theta_2).$$

(ii) If $\rho_p(\theta_1) \le \sigma_q(\theta_1)$ and $x_p(\theta_2) = y_q(\theta_2)$, then starting from the time θ_2 we will consider the pursuit game involving $(p-1)$ pursuers and $(q-1)$ evaders under the condition

$$\rho_1^2(\theta_2) + \cdots + \rho_{p-1}^2(\theta_2) > \sigma_1^2(\theta_2) + \cdots + \sigma_{q-1}^2(\theta_2).$$

(iii) If $\rho_p(\theta_1) \le \sigma_q(\theta_1)$ and $x_p(\theta_2) \neq y_q(\theta_2)$, then starting from the time θ_2 we will consider the pursuit game involving $(p-1)$ pursuers and q evaders under the condition

$$\rho_1^2(\theta_2) + \cdots + \rho_{p-1}^2(\theta_2) > \sigma_1^2(\theta_2) + \cdots + \sigma_q^2(\theta_2),$$

and so on. Applying this procedure repeatedly allows us to obtain eventually an inequality $\rho_s^2(\theta_j) > \sigma_r^2(\theta_j)$ for some $s \in \{1, \dots, m\}$, $r \in \{1, \dots, k\}$. Then, clearly, the pursuer x_s completes the pursuit for some finite time T. The proof of theorem is complete. □

4 Conclusion

In this paper, we have investigated a simple motion pursuit differential game of m pursuers and k evaders on a cylinder. We have reduced the differential game to a differential game of m groups of countably many pursuers and k groups of countably many evaders in $\mathbb{R}^2$. All the players from each group are controlled by one control parameter subject to integral constraint. Pursuit is completed when the position of a pursuer coincides with that of the evader. We have evinced that if the total resource of the pursuers is greater than that of the evaders, then pursuit can be completed. We constructed the strategies for the pursuers.

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Conflicts of Interest The authors declare no conflict of interest.

References

[1] M. Altaher, S. Elmougy & O. Nomir (2019). Intercepting a superior missile: a reachability analysis of an apollonius circle-based multiplayer differential game. *International Journal of Innovative Computing Information and Control*, *15*(1), 369–381.

[2] A. A. Azamov & A. S. Kuchkarov (2009). Generalized lion & man game of R. Rado. *Contributions to Game Theory and Management*, *2*, 8–20.

[3] M. Ferrara, G. Ibragimov, I. A. Alias & M. Salimi (2019). Pursuit differential game of many pursuers with integral constraints on compact convex set. *Bulletin of the Malaysian Mathematical Sciences Society*, *43*, 2929–2950. https://doi.org/10.1007/s40840-019-00844-3.

[4] G. Ibragimov & N. Satimov (2012). A multiplayer pursuit differential game on a closed convex set with integral constraints. *Abstract and Applied Analysis*, *2012*, Article ID: 460171, 12 pages. https://doi.org/10.1155/2012/460171.

[5] G. I. Ibragimov & R. M. Hasim (2010). Pursuit and evasion differential games in hilbert space. *International Game Theory Review*, *12*(3), 239–251.

[6] R. Isaacs (1999). *Differential games: a mathematical theory with applications to warfare and pursuit, control and optimization*. Courier Corporation, North Chelmsford, Massachusetts.

[7] R. P. Ivanov (1980). Simple pursuit-evasion on a compactum. In *Doklady Akademii Nauk*, pp. 1318–1321. Russian Academy of Sciences, Moscow, Russia.

[8] A. S. Kuchkarov (2009). A simple pursuit-evasion problem on a ball of a riemannian manifold. *Mathematical Notes*, *85*(1-2), 190–197.

[9] A. S. Kuchkarov, R. M. Hasim & M. A. Hassan (2011). Differential games with many pursuers when evader moves on the surface of a cylinder. *ANZIAM Journal*, *53*, 1–20. https://doi.org/10.21914/anziamj.v53i0.3280.

[10] A. A. Melikyan & N. V. Ovakimyan (1993). A differential game of simple approach in manifolds. *Journal of Applied Mathematics and Mechanics*, *57*(1), 47–57.

[11] A. Melikyan (1998). Games of simple pursuit and approach on two-dimensional cone. In *Generalized Characteristics of First Order PDEs*, pp. 169–198. Springer, Boston, MA. https://doi.org/10.1007/978-1-4612-1758-9_6.

[12] V. V. Nikulin & I. R. Shafarevich (1987). *Geometries and groups*. Springer Science & Business Media, Nauka, Moscow.

[13] L. A. Petrosjan (1993). *Differential games of pursuit*. World Scientific, St. Petersburg, Russia.

[14] N. N. Petrov (1997). Simple pursuit of rigidly linked evaders. *Autom Remote Control*, *58*(12), 1914–1919.

[15] N. N. Petrov & K. A. Shchelchkov (2014). About the problem of group persecution in linear differential games with a simple matrix and state constraints. *International Journal of Pure and Applied Mathematics*, *92*(1), 13–26. http://dx.doi.org/10.12732/ijpam.v92i1.2.

[16] B. N. Pshenichnyi (1976). Simple pursuit by several objects. *Cybernetics and Systems Analysis*, *12*(3), 484–485.

[17] M. V. Ramana & M. Kothari (2017). Pursuit-evasion games of high speed evader. *Journal of Intelligent & Robotic Systems*, *85*(2), 293–306.

[18] N. Satimov, B. B. Rikhsiev & A. A. Khamdamov (1983). On a pursuit problem for n-person linear differential and discrete games with integral constraints. *Mathematics of the USSR-Sbornik*, *46*(4), 459–471.